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FINITE DIMENSIONAL POLISH SPACES ARE EXTREME BOUNDARIES OF CONVEX BODIES IN EUCLIDEAN SPACE

BY

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ABSTRACT

It is proved that every n-dimensional Polish space is homeomorphic to the set of extreme points of a compact convex set in $R^{18(n+1)}$.

In [L] A. Lazar proved that every uncountable Polish space (= complete metrizable separable space) is homeomorphic to the set of extreme points of some weakly compact convex body in Hilbert space. Later, he raised the following problem: Does there exist a function $k = k(n)$ such that every n-dimensional Polish space is homeomorphic to the set of extreme points of some compact convex body in the Euclidean space R^k?

In this article we give an affirmative answer to this problem of Lazar.

It should be noted that there is an essential difference between Lazar's result and ours. In the infinite dimensional case the main problem was to force the weakly compact convex set under consideration to be a convex body (i.e. to have a non-empty interior). It was known earlier [H] that every Polish space is homeomorphic to the extreme boundary of some compact convex set, and also [L–P] that the extreme boundary of a convex body in an infinite dimensional reflexive Banach space is uncountable. In the finite dimensional case the main problem was to construct any suitable compact convex set, which will then automatically be a convex body in the minimal hyperplane that contains it.

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We state now three theorems, the last of which settles Lazar's problem while the first two provide stronger results under some additional hypothesis. Through this article X will denote a non-compact Polish space. (Lazar's problem is trivial if X is compact. If $\dim X \leq n$ then (see 5, below) there exists a homeomorphism h of X into the $(2n+1)$-dimensional Euclidean sphere S_{2n+1} in R^{2n+2}. Set $K = \overline{\mathrm{conv}}\, h(X)$. As S_{2n+1} is strictly convex, if X is compact then $h(X) = \mathrm{ext}\, K$ (= the set of extreme points of K).) If $W = X \cup F$ is a compactification of X then we assume that $F = W \setminus X$. All spaces are assumed to be separable metric, and dim refers to the topological dimension.

THEOREM 1. *Let X be an n-dimensional Polish space and let $W = X \cup F$ be a compactification of X with* $\dim F = 0$. *Then there exist a compact convex set K in R^{2n+5} and a homeomorphism $h: W \to K$ such that $h(X) = \mathrm{ext}\, K$.*

THEOREM 2. *Let X be a perfect Polish space and let W be an n-dimensional compactification of X. Then there exist a compact convex set K in $R^{18(n+1)}$ and a homeomorphism $h: W \to K$ such that $h(X) = \mathrm{ext}\, K$.*

THEOREM 3. *Every n-dimensional Polish space is homeomorphic to the set of extreme points of a compact convex set in $R^{18(n+1)}$.*

Note that in Theorems 1 and 2, X is given with a compactification W and the homeomorphism $h: X \to \mathrm{ext}\, K$ is extendable over W (into K). We pay for this by the assumptions that $\dim F = 0$ in Theorem 1 and that X is perfect in Theorem 2. We could not prove a similar result in the general case, and leave it as a problem.

PROBLEM. Let W be an n-dimensional compactification of Polish space X. Do there exist a compact convex set K in R^k and a homeomorphism $h: W \to K$ so that $h(X) = \mathrm{ext}\, K$? Can k depend only on n? In particular does this hold if X is countable? (The main problem is when X is countable and $\dim(W \setminus X) \geq 1$.)

The following are some dimension-theoretic results that will be applied in the proofs. (Refer to [E] for those stated without a proof.)

(1) A family of sets is of order n if no $n+2$ of its elements have a point in common.

(2) A compact n-dimensional space admits a finite closed cover of order n and arbitrary small diameter.

(3) An n-dimensional space admits an n-dimensional compactification.

(4) A zero-dimensional Polish space X admits a zero-dimensional compactification $W = X \cup F$ with F countable.

PROOF. By Th. 2, p. 441 in [K] X is homeomorphic to a *closed* subset of the irrationals, which, in turn, can be realized as a subset of the Cantor set C with a countable complement. Now take W to be the closure of X in C. □

(5) CLASSICAL EMBEDDING THEOREM. *An n-dimensional space W is homeomorphic to a subset of* R^{2n+1}. *Moreover, the embeddings in* $C(W, R^{2n+1})$ *are complemented by a set of first category.*

(6) A THEOREM OF HUREWICZ. *Let* $f: W \to Z$ *be a closed continuous surjection. Then* $\dim Z \leq \dim W + \operatorname{card} f - 1$ *where* $\operatorname{card} f = \sup\{\operatorname{card} f^{-1}(z) : z \in Z\}$.

(7) A kind of an inverse to 6: *Let F be n-dimensional and compact. Then there exist a zero-dimensional compact space V and a mapping f of V onto F with* $\operatorname{card} f \leq n + 1$.

(8) Let F be σ-compact. Then F admits a countable compact cover of order ≤ 1.

PROOF. Let $F = \bigcup_{i \geq 1} H_i$ be any compact cover, with $H_1 \subset H_2 \subset \cdots$.
Let $f_i : F \to [0,1]$ be continuous with $f_i^{-1}(0) = H_i$.
Set $I_j = [1/(j+1), 1/j]$ and $F_{i,j} = f_i^{-1}(I_j) \cap H_{i+1}$.
Then $F = H_1 \cup \bigcup_{i,j} F_{i,j}$ and the order of this cover is ≤ 1. □

(9) $\dim(A \cup B) \leq \dim A + \dim B + 1$.

The following lemma is our main tool for proving the theorems.

MAIN LEMMA. *Let* $W = X \cup F$ *be a compactification of X. Let Y be compact, let* $\pi : W \to Y$ *be a surjection and let* $l : W \to R^m$ *be a mapping so that for each y in Y*

(i) l *is one-to-one on* $\pi^{-1}(y)$,

(ii) $l(\pi^{-1}(y) \cap X)$ *is the extreme boundary of* $\overline{\operatorname{conv}}\, l(\pi^{-1}(y))$.

If $f: Y \to S_k \subset R^{k+1}$ *is an embedding, then the mapping* $h = (f \circ \pi, l) : W \to R^{k+1} \times R^m$ *is an embedding and* $h(X) = \operatorname{ext} \overline{\operatorname{conv}}\, h(W)$.

PROOF. h is one-to-one since the fibers of $f \circ \pi$ are those of π (as f is one-to-one) and by (i) l is one-to-one on $\pi^{-1}(y)$. Set $K = \overline{\operatorname{conv}}\, h(W)$. Clearly $\operatorname{ext} K \subset h(W)$. Note first that if $x_0 \in F$ then $h(x_0)$ is not an extreme point of K. Indeed let

$$h(x_0) = (w_1, w_2) = (f(\pi(x_0)), l(x_0)) \in R^{k+1} \times R^m.$$

By (ii) $l(\pi^{-1}(f^{-1}(w_1)) \cap X)$ is the extreme boundary of $\overline{\operatorname{conv}}\, l(\pi^{-1}(f^{-1}(w_1)))$ in R^m. It follows that $w_2 = l(x_0)$ is a non-trivial convex combination of some points $l(x_1), l(x_2), \ldots, l(x_r)$ in $l(\pi^{-1}(f^{-1}(w_1)) \cap X)$ with $\{x_i\}_{i=1}^r \subset X$. As $h(x_i) = (w_1, l(x_i))$, $1 \leq i \leq r$ and $h(x_0) = (w_1, l(x_0))$, $h(x_0)$ is the same convex combination of $h(x_1),\dots,h(x_r)$. Thus $\operatorname{ext} K \subset h(X)$. Let $x_0 \in X$. If $h(x_0)$ is not in $\operatorname{ext} K$ then it must be a non-trivial convex combination of some points $h(x_1)$, $\dots,h(x_r)$, $\{x_i\}_{i=1}^r \subset X$ $(x_i \neq x_0)$. Then in particular $f(\pi(x_0))$ is a convex combination, with the same non-trivial coefficients, of $f(\pi(x_i))$, $1 \le i \le r$. But as $f(Y)$ is contained in the sphere $S_k \subset R^{k+1}$ which is strictly convex, this is possible only if

$$f(\pi(x_0)) = f(\pi(x_1)) = \cdots = f(\pi(x_r)) = w_1.$$

So $\{x_i\}_{x=0}^r \subset \pi^{-1}(y) \cap X$ for $y = f^{-1}(w_1) \in Y$, and $l(x_0)$ is a non-trivial convex combination of $l(x_1),\dots,l(x_r)$ which violates (ii). □

Example. If X is locally compact and $W = X \cup F$ is a compactification, then F is compact. Let $\dim X = n_1$ and $\dim F = n_2$. Let $F' \subset X$ consist of $2n_2 + 2$ points. Let $l: W \to R^{2n_2+1}$ be such that $l/F \cup F'$ is an embedding and $\operatorname{conv} l(F')$ is a simplex which contains $l(F)$ in its interior. Let Y be the quotient of W obtained by identifying $F \cup F'$ to a single point, and let $\pi: W \to Y$ be the quotient map. Then $\dim Y = \dim X$ (since Y is obtained by adding one point to $X \setminus F'$). Let $f: Y \to S_{2n_1+1} \subset R^{2n_1+2}$ be an embedding. It is easy to check that the conditions of the Main Lemma are satisfied. Hence $h = (f \circ \pi, l): W \to R^{2(n_1+n_2)+3}$ is an embedding which maps X onto $\operatorname{ext} \overline{\operatorname{conv}}\, h(W)$. Similar ideas are used in the proof of the Theorems.

Proof of Theorem 1. Let $F = \bigcup_{i \ge 1} F_i$ with F_i compact order $\{F_i\} = 0$ and $\lim_{i \to \infty} \operatorname{diam} F_i = 0$.

Let U be a neighborhood of some F_i in W. Set

$$D_{U,i} = \{f : f \in C(W)\text{: there exist points } a_i \text{ and } b_i \text{ in } X \cap U \text{ such that } f(a_i) < \min f(F_i) \le \max f(F_i) < f(b_i)\}.$$

By standard arguments $D_{U,i}$ is open and dense in $C(W)$.

Let U_k^i be a $1/k$ neighborhood of F_i in W. From the Baire category theorem and (5), it follows that there exists some

$$l \in \bigcap_{\substack{i \ge 1\\ k \ge 1}} D_{U_k^i,i},$$

so that l is one-to-one on each F_i, $i \ge 1$. As

$$l \in \bigcap_{\substack{i \ge 1\\ k \ge 1}} D_{U_k^i,i}$$

we may select inductively disjoint pairs $\{a_i, b_i\}_{i\geq 1}$ of points in X so that

$$l(a_i) < \min l(F_i) \leq \max l(F_i) < l(b_i), \quad i \geq 1 \quad \text{and} \quad \lim_{i\to\infty} \operatorname{diam}(F_i \cup \{a_i, b_i\}) = 0.$$

Consider the decomposition of W into closed sets of the form $F_i \cup \{a_i, b_i\}$, $i \geq 1$, and the singletons of $X \setminus \bigcup_{i\geq 1} \{a_i, b_i\}$. Every non-trivial limit of a sequence of elements of this decomposition must be a singleton. It follows that the decomposition is uppersemicontinuous. Let Y denote the quotient space and $\pi: W \to Y$ the quotient map corresponding to this decomposition. Then

$$Y = \pi\Bigl(\bigcup_{i\geq 1} \{F_i \cup \{a_i, b_i\}\}\Bigr) \cup \pi\Bigl(X \setminus \bigcup_{i\geq 1} \{a_i, b_i\}\Bigr).$$

The first set in this union is countable and hence zero-dimensional while the second is homeomorphic (by π^{-1}) to a subset of X so its dimension does not exceed n. If follows (by (9)) that $\dim Y \leq 0 + n + 1 = n + 1$. Thus, there exists a homeomorphism $f: Y \to S_{2n+3} \subset R^{2n+4}$, and it is easy to verify that the conditions of the Main Lemma are satisfied. Hence

$$h = (f \circ \pi, l): W \to R^{2n+4} \times R = R^{2n+5} \quad \text{and} \quad K = \overline{\operatorname{conv}}\, h(W)$$

have the desired properties. □

Proof of Theorem 2. Let $W = X \cup F$. We shall construct a zero-dimensional subset V of X and a continuous surjection $g: V \to F$ with the following properties:

(i) The decomposition of W into sets of the form $\{x\} \cup g^{-1}(x)$, $x \in F$ and $\{x\}$, $x \in X \setminus V$ is uppersemicontinuous.

(ii) The dimension of the quotient space Y obtained by the above decomposition does not exceed $7n + 6$.

(iii) There exists a mapping $l: X \to R^{4n+4}$ such that, for every $x \in F$, l is one-to-one on $\{x\} \cup g^{-1}(x)$, and the extreme points of $\operatorname{conv}\{l(\{x\} \cup g^{-1}(x))\}$ are exactly the elements of $l(g^{-1}(x))$; in particular $l(x)$ is not an extreme point of this set.

Once we accomplish this we are done. Indeed, as there exists a homeomorphism $f: Y \to S_{14n+13} \subset R^{14(n+1)}$, and since the conditions of the Main Lemma are satisfied, it follows that

$$h = (f \circ \pi, l): W \to R^{14(n+1)} \times R^{4n+4} = R^{18(n+1)} \quad \text{and} \quad K = \overline{\operatorname{conv}}\ h(W)$$

have the desired properties.

The mapping l will be defined to be the limit of a sequence $\{l_i\}_{i\geq 0}$ of mappings which, together with the set V and the function g, will be defined inductively as follows:

By (8) $F = \bigcup_{i\geq 1} F_i$ with F_i compact and order $\{F_i\} \leq 1$.

Let $\{M_i\}_{i\geq 1}$ be a sequence of two-dimensional linear subspaces of R^{4n+4} with the following property:

(*) For every set $\{i_1, i_2, \ldots, i_{2n+2}\}$ of $2n+2$ distinct indices, if $0 \neq y_{i_j} \in M_{i_j}$ then $\{y_{i_j}\}_{j=1}^{2n+2}$ is linearly independent in R^{4n+4}.

Set $l_0 \equiv (0,0,\ldots,0) \in R^{4n+4}$, and $V_0 = \varnothing$. Assume that we have already constructed compact disjoint subsets $V_1, V_2, \ldots, V_i$ of x, with an empty interior in W, and continuous mappings $l_j : W \to R^{4n+4}$, $0 \leq j \leq i$ and $g : \bigcup_{j=0}^{i} V_j \to \bigcup_{j=1}^{i} F_j$.

By (7) there exist a zero-dimensional compact space T and a continuous surjection $\varphi : T \to F_{i+1}$, with card $\varphi \leq n+1$. Let $T = \bigcup_s T_s$ be a finite closed cover of T so that order $\{T_s\} = 0$ diam $H_s < 1/(i+1)$ and diam $l_i(H_s) < 1/8^i$ where $H_s = \varphi(T_s)$. Then $F_{i+1} = \bigcup_s H_s$ and order $\{H_s\} \leq n$. As X is perfect, there exists, for each s, a set $C_s \subset X \backslash \bigcup_{j=1}^{i} V_j$ which is homeomorphic to the Cantor set, whose interior in W is empty, so that $\operatorname{diam}(C_s \cup H_s) < 1/(i+1)$ and diam $l_i(C_s \cup H_s) < 1/8^i$, and such that the sets $\{C_s\}$ are mutually disjoint. For each s, let Z_s^1, Z_s^2 and Z_s^3 be three disjoint copies of T_s in C_s. Set $Z_s = \bigcup_{r=1}^{3} Z_s^r$ and $V_{i+1} = \bigcup_s Z_s$. Extend g to V_{i+1} by defining it to be equal to φ/T_s on each of its copies Z_s^r, $r = 1,2,3$. It follows that for x in V_{i+1} diam $\{x, g(x)\} < 1/(i+1)$. Also, since order $\{F_i\} \leq 1$, order $\{H_s\} \leq n$ and, as $1 \leq r \leq 3$, card $g \leq 2\cdot 3(n+1) = 6(n+1)$.

Now we define l_{i+1}. For each s let M_s be one of the two-dimensional subspaces of R^{4n+4} that satisfy (*) so that M_s has not been selected earlier in the inductive process, and so that the subspaces M_s are distinct. For a fixed value of s, let $R^{4n+4} = M_s \oplus M_s^{\perp}$, let p_s denote the orthogonal projection of R^{4n+4} onto M_s and q_s the one on $M_s^{\perp}$. As diam $l_i(H_s) < 1/8^i$, diam $p_s\, l_i(H_s) < 1/8^i$ too. Let $S \subset M_s$ be a circle of diameter $1/8^{i-1}$ with center at one of the points of $p_s l_i(H_s)$. Let S^1, S^2 and S^3 be three disjoint arcs on S so that for every choice of points $a_r \in S^r$, $r = 1,2,3$, the triangle conv$\{a_1, a_2, a_3\}$ contains $p_s l_i(H_s)$.

Let $\psi_s : \bigcup_{r=1}^{3} Z_s^r \to M_s$ map Z_s^r homeomorphically into S^r. Define l_{i+1} on $Z_s = \bigcup_{r=1}^{3} Z_s^r$ by $l_{i+1}(x) = (\psi_s(x), q_s l_i g(x)), x \in Z_s$ (i.e., $p_s l_{i+1}(x) = \psi_s(x)$ and $q_s l_{i+1}(x) = q_s l_i g(x)$). Note that $\| l_{i+1} - l_i \|_{Z_s} \leq 2/8^{i-1}$. Indeed, in M_s

$$\| P_s l_{i+1}(x) - P_s l_i(x) \| = \| \psi_s(x) - P_s l_i(x) \| < \frac{1}{8^{i-1}}$$

since $\psi_s(x) \in S$ while $P_s l_i(x)$ is in the interior of S as diam $l_i(C_s \cup H_s) < 1/8^i$.

In $M_s^\perp$

$$\|q_s l_{i+1}(x) - q_s l_i(x)\| = \|q_s l_i g(x) - q_s l_i(x)\|$$

$$\leq \|l_i g(x) - l_i(x)\| \leq \operatorname{diam} l_i(C_s \cup H_s) < \frac{1}{8^i}$$

since for $x \in Z_s$, $\{x, g(x)\} \subset Z_s \cup H_s \subset C_s \cup H_s$. In this way we define l_{i+1} on each Z_s, i.e. on V_{i+1}, such that $\|l_{i+1} - l_i\|_{V_{i+1}} < 2/8^{i-1}$. Define also l_{i+1} to equal l_i on $\bigcup_{j=1}^{i} V_j \cup \bigcup_{j=1}^{i+1} F_j$, and extend it to a mapping $l_{i+1}: W \to R^{4n+4}$ so that $\|l_{i+1} - l_i\| < 2/8^{i-1}$. This concludes the inductive process.

Set $V = \bigcup_{i\geq 1} V_i$, $l = \lim_{i\to\infty} l_i$ and let $g: V \to F$ be the map defined in the inductive process. Let us check that (i), (ii) and (iii) hold.

(i) To show uppersemicontinuity it suffices to prove that if $\{A_r\}_{r=1}^{\infty}$ is a sequence of elements of the decomposition which converges to some subset A of W (in the Hausdorff metric on W, say) then A is contained in some element of the decomposition. So let $\{A_r\}_{r\geq 1}$ be such a sequence. If A_r is a singleton for infinitely many values of r, then so is A and we are done. Thus we may assume that $A_r = \{x_r\} \cup g^{-1}(x_r)$, $x_r \in F$. If there exists some i so that $x_r \in F_i$ for infinitely many values of r, then, by the compactness of F_i, we may assume that x_r converges to some element x_0 of F_i. As order $\{F_i\} \leq 1$, each x_r may belong to at most one more set F_j. If, for some value of j, F_j contains infinitely many x_r's, then as $g: V_i \cup V_j \to F_i \cup F_j$ is a continuous mapping of compact spaces, and since for each $r \geq 0$, $g^{-1}(x_r) \subset V_i \cup V_j$, A must be contained in $\{x_0\} \cup g^{-1}(x_0)$. If such a j does not exist then

$$\lim_r \operatorname{diam}(g^{-1}(x_r) \cap (W \setminus V_i)) = 0,$$

and, since $g: V_i \to F_i$ is a continuous mapping of compact spaces, $A \subset \{x_0\} \cup g^{-1}(x_0)$. If, finally, such an i fails to exist, then, since $\operatorname{diam}\{\{x\} \cup g^{-1}(x)\} < 1/(i+1)$ for $x \in F_i \setminus \bigcup_{j=1}^{i-1}$, A must be a singleton and (i) follows.

(ii) Let $\pi: W \to Y$ be the quotient map. As the elements of the decomposition are either singletons or of the form $\{x\} \cup g^{-1}(x)$, $x \in F$, it follows that card π = card $g + 1 \leq 6(n+1) + 1$. Hence from the Hurewicz Theorem (6) we obtain

$$\dim Y \leq \dim W + \operatorname{card} \pi - 1 \leq n + 6(n+1) + 1 - 1 = 7n + 6.$$

(iii) Let $x \in F$. We check first that $l(x)$ is not in ext L where $L = \operatorname{conv}\{l(\{x\} \cup g^{-1}(x))\}$. Let $x \in H_s \subset F_i \subset F$. Recall that by the inductive construction there exist three points $x_r \in Z_s^r$, $r = 1,2,3$ with $g(x_r) = x$, and such that $p_s l$ maps x_r to some point $p_s l(x_r) = a_r$ in $S^r \subset S$ (where p_s and q_s are the projections of R^{4n+4} on M_s and $M_s^\perp$, respectively). From the choice of the arcs S^r it follows that $p_s l(x)$ is a nontrivial convex combination of $a_r = p_s l(x_r)$, $r = 1,2,3$ in M_s. Also, as by the definition of l, $q_s l(x_r) = q_s lg(x_r) = q_s(x)$, $l(x)$ is a convex combination of $l(x_r)$, $r = 1,2,3$ (with the same coefficients) and $l(x)$ is not an extreme point of L.

Next we show that for each $y \in g^{-1}(x)$, $l(y)$ is an extreme point of L. We may assume that $l(x) = 0$ in R^{4n+4}. (If not, then we apply a translation.) Let $w \in g^{-1}(x)$. Then $w \in Z_s$ for some s such that $x \in H_s$. By the definition of l we have

$$l(w) = p_s l(w) + q_s l(w) = p_s l(w) + q_s lg(w) = p_s l(w) + q_s l(x)$$
$$= p_s l(w) + q_s(0) = p_s l(w) \in M_s.$$

(Note that $l(w)$ is on a circle of positive radius in M_s and hence differs from $0 = l(x)$.)

If, for $y \in g^{-1}(x)$, $l(y)$ is not an extreme point of L, then it is a convex combination of points $l(w)$, $w \in g^{-1}(x)$. Each $x \in F$ is in H_s for at most $2(n+1)$ values of s (since order $\{F_i\} \leq 1$ and order $\{H_s\} \leq n$). It follows that $g^{-1}(x)$ is contained in the union of at most $2(n+1)$ sets Z_s, and hence $lg^{-1}(x)$ is contained in the union of M_s for at most $2(n+1)$ values of s. From the property (*) of the subspaces M_s it follows that the convex combination that represents $l(y)$ is actually supported in $\{l(w) : w \in Z_{s_0}\}$, where s_0 is the index such that $y \in Z_{s_0}$. But l maps Z_{s_0} homeomorphically into a circle in M_{s_0}, which is strictly convex. It follows that the convex combination is actually trivial and we are done. The fact that l is one-to-one on $\{x\} \cup g^{-1}(x)$ follows from the above.

This completes the proof of Theorem 2. □

PROOF OF THEOREM 3. Let X be an n-dimensional Polish space. Set $X_0 = \{x \in X : x$ has a countable neighborhood in $X\}$ and $X_1 = X \setminus X_0$. Then X_0 is a countable open subset of X and X_1 is closed and perfect. Both X_0 and X_1 are Polish spaces. If $\dim X = 0$, then we apply (3) to find a zero-dimensional compactification for X and conclude by Theorem 1. So, we assume that $n = \dim X = \dim X_1 \geq 1$. Let Y be an n-dimensional compactification of X. Let M denote the closure of X_1 in Y. As X_0 is open in X, $X_0 \cap M = \emptyset$, and $M = X_1 \cup F_1$ is a compactification of X_1. Let Y_0 be the quotient of Y obtained by identifying the elements of M with a single point m. The subspace $X_0 \cup \{m\}$ of Y_0 is a countable Polish space. Hence, by (4) it admits a countable compactification $Z = \{m\} \cup X_0 \cup F_0$. Consider the disjoint union A of $A_0 = X_0 \cup F_0 \subset Z$ and $A_1 = X_0 \cup M = X_0 \cup X_1 \cup F_1 \subset Y$. ($U \subset A$ is open if both $U \cap A_1$ and $U \cap A_2$ are open in A_1 and A_2, respectively.) Let W be the quotient of A obtained by identifying the corresponding points of X_0 in A_0 and A_1. (Formally, if $\mathrm{id}:(X_0 \subset A_0) \to (X_0 \subset A_1)$ is the identity map, W is obtained by identifying the pairs $\{x, \mathrm{id}\,x\}_{x \in X_0 \subset A_0}$ to a single point.) It is easy to check that W is a compactification of X (check for sequential compactness) which we regard as $W = F_0 \cup X_0 \cup X_1 \cup F_1$. Also, as $M = X_1 \cup F_1$ is compact and $F_0 \cup X_0$ is countable, $\dim W = \dim M = n$. The main advantage that the present compactification W of X has upon the earlier one Y is that, in W, the "contribution" F_0 of X_0 to the compactification is countable. We could have managed with a zero-dimensional F_0 as well. But in Y we had no control on the dimension of the "contribution" of X_0 to the compactification.

Operate now on $M = X_1 \cup F_1$ as in the proof of Theorem 2. We obtain a subset V_1 of X_1, $g_1 : V_1 \to F_1$ and $l_1 : M \to R^{4n+4}$ as in Theorem 2. Let $R^{4n+4} = R \times R^{4n+3}$ and let p and q denote the projections onto R and R^{4n+3}, respectively. Let $F_0 = \{a_i\}_{i \geq 1}$. Applying arguments similar to those in the proof of Theorem 1 we find an extension $\psi : W \to R$ of $pl_1 : M \to R$ and distinct points $\{b_i, c_i\}_{i \geq 1}$ in X_0 with $\lim_i \operatorname{diam}\{a_i, b_i, c_i\} = 0$ and so that $\psi(b_i) < \psi(a_i) < \psi(c_i)$. Next, we find an extension $\varphi : W \to R^{4n+3}$ of $ql_1 : M \to R^{4n+3}$ so that $\varphi(a_i) = \varphi(b_i) = \varphi(c_i)$, $i \geq 1$. (Let $W_0 = W$ modulo $a_i = b_i = c_i$, $i \geq 1$, let $\pi : W \to W_0$ be the quotient, let $\varphi_1 : W_0 \to R^{4n+3}$ be any extension of pl_1 (note that $M \subset W_0$) and take $\varphi = \varphi_1 \pi$.) Let $l : W \to R^{4n+4}$ be such that $pl = \psi$ and $ql = \varphi$. Then l is an extension of l_1. Let also $V = V_1 \cup \{b_i\}_{i \geq 1} \cup \{c_i\}_{i \geq 1}$ and extend $g_1 : V_1 \to F_1$ to $g : V \to F = F_0 \cup F_1$ by defining $g(b_i) = g(c_i) = a_i$, $i \geq 1$.

As in the proof of Theorems 1 and 2, the decomposition of W into $\{x\} \cup g^{-1}(w)$, $x \in F$ and the singletons of $X \setminus V$ is upper semicontinuous. Let Y and $\pi : W \to Y$ be the corresponding quotient space and quotient map. As $\operatorname{card} g = \operatorname{card} g_1$, it follows from (6) that $\dim Y \leq 7n + 6$. It is easy to check that the conditions of the Main Lemma hold, and the Theorem follows. □

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